

# $\lambda$ -symmetry criteria for linearization of second order ODEs via point transformations

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## Abstract

An alternative proof of Lie's approach for linearization of scalar second order ODEs is derived using the relationship between  $\lambda$ -symmetries and first integrals. This relation further leads to a new  $\lambda$ -symmetry linearization criteria for second order ODEs which provides a new approach for constructing the linearization transformations with lower complexity. The effectiveness of the approach is illustrated by obtaining the local linearization transformations for the linearizable nonlinear ODEs of the form  $y'' + F_1(x, y)y' + F(x, y) = 0$ . Examples of linearizing nonlinear ODEs which are quadratic or cubic in the first derivative are also presented.

Key words: Lie's linearization, second order ODEs, point transformations,  $\lambda$ -symmetries.

# 1 Introduction

The initial seminal studies of scalar second-order ordinary differential equations (ODEs) which are linearizable by means of point transformations are due to Lie [1] and Tressé [2]. In recent decades there have been a resurgence of interest in this topic (see the reviews [3, 4, 5]).

It was shown by Lie [1] that any second-order ODE

$$y'' = f(x, y, y') \quad (1.1)$$

which is linearizable via point transformations is at most cubic in the first derivative, i.e. it has the form

$$y'' + F_3(x, y)y'^3 + F_2(x, y)y'^2 + F_1(x, y)y' + F(x, y) = 0. \quad (1.2)$$

It is well known [6] that any second order linear ODE can be transformed via point transformations to the free particle equation

$$u_{tt} = 0. \quad (1.3)$$

Therefore, all linearizable second-order ODEs (1.1) are obtained from the free particle equation (1.3) via point transformations. Precisely, the free particle equation (1.3) can be transformed by an arbitrary change of variables

$$t = \phi(x, y), \quad u = \psi(x, y), \quad J = \phi_x \psi_y - \phi_y \psi_x \neq 0, \quad (1.4)$$

where  $J$  is the Jacobian, to the family of ODEs (1.2) with the coefficients  $F(x, y)$ ,  $F_1(x, y)$ ,  $F_2(x, y)$  and  $F_3(x, y)$  satisfying the following system of partial differential equations

$$F_3(x, y) = A, \quad F_2(x, y) = B + 2w, \quad F_1(x, y) = P + 2z, \quad F(x, y) = Q, \quad (1.5)$$

in which

$$\begin{aligned} A &= \frac{\phi_y \psi_{yy} - \psi_y \phi_{yy}}{\phi_x \psi_y - \phi_y \psi_x}, & B &= \frac{\phi_x \psi_{yy} - \psi_x \phi_{yy}}{\phi_x \psi_y - \phi_y \psi_x}, & w &= \frac{\phi_y \psi_{xy} - \psi_y \phi_{xy}}{\phi_x \psi_y - \phi_y \psi_x}, \\ Q &= \frac{\phi_x \psi_{xx} - \psi_x \phi_{xx}}{\phi_x \psi_y - \phi_y \psi_x}, & z &= \frac{\phi_x \psi_{xy} - \psi_x \phi_{xy}}{\phi_x \psi_y - \phi_y \psi_x}, & P &= \frac{\phi_y \psi_{xx} - \psi_y \phi_{xx}}{\phi_x \psi_y - \phi_y \psi_x}. \end{aligned} \quad (1.6)$$

Lie [1] crucially also found the following over-determined system of four equations

$$\begin{aligned} w_x &= zw - F F_3 - \frac{1}{3} \frac{\partial F_1}{\partial y} + \frac{2}{3} \frac{\partial F_2}{\partial x}, \\ w_y &= -w^2 + F_2 w + F_3 z + \frac{\partial F_3}{\partial x} - F_1 F_3, \\ z_x &= z^2 - F_1 z - F w + \frac{\partial F}{\partial y} + F F_2, \\ z_y &= -zw + F F_3 - \frac{1}{3} \frac{\partial F_2}{\partial x} + \frac{2}{3} \frac{\partial F_1}{\partial y}, \end{aligned} \quad (1.7)$$

which are called the Lie conditions. The compatibility of Lie's conditions give the following well known Lie-Tressé linearization test for ODEs of the form (1.2), viz.

$$\begin{aligned} \frac{\partial^2 F_1}{\partial y^2} - 2 \frac{\partial^2 F_2}{\partial y \partial x} + 3 \frac{\partial^2 F_3}{\partial x^2} - 3 \frac{\partial F_1}{\partial x} F_3 - 3 F_1 \frac{\partial F_3}{\partial x} + 6 \frac{\partial F}{\partial y} F_3 + 3 F \frac{\partial F_3}{\partial y} - F_2 \frac{\partial F_1}{\partial y} + 2 F_2 \frac{\partial F_2}{\partial x} &= 0 \\ \frac{\partial^2 F_2}{\partial x^2} - 2 \frac{\partial^2 F_1}{\partial y \partial x} + 3 \frac{\partial^2 F}{\partial y^2} + 3 \frac{\partial F}{\partial y} F_2 + 3 F \frac{\partial F_2}{\partial y} - 3 \frac{\partial F}{\partial x} F_3 - 6 F \frac{\partial F_3}{\partial x} + F_1 \frac{\partial F_2}{\partial x} - 2 F_1 \frac{\partial F_1}{\partial y} &= 0. \end{aligned} \quad (1.8)$$

It was Tressé [2] who first obtained the invariant criteria (1.8).

We point out that the conditions (1.8) can also be deduced via the Cartan equivalence approach [7]. More recently in two independent papers [8, 9], the authors present geometrical proofs for the determination of the invariant conditions (1.8). It should be noted that the projections utilized in [8, 9] were different to each other. In [10] Lie's linearizability criteria were utilized to determine a linearizable class of a system of two second order equations that are obtainable from complex scalar second order ODEs. Furthermore, Mahomed and Qadir [11] found criteria for conditional linearizability of third order ODEs via point transformation subject to a Lie-Tressé linearizable second order ODE.

One can mention here as well the results on the algebraic criteria for linearization by point transformations of scalar second order ODEs (1.1). It is well known from Lie's work that such linearizable ODEs possess eight point symmetries. The question arises if one can conclude linearization of (1.1) in case one has knowledge of fewer than eight symmetries

of the ODE (1.1). The answer is affirmative. In fact Lie was the first to obtain algebraic criteria for linearization when the ODE (1.1) admits two symmetries which are connected. Some further studies [12, 13] provided input on the algebraic criteria for linearization when the ODE (1.1) has two unconnected symmetries.

Lie proved that the ODE (1.1) is linearizable by point transformations if and only if the over-determined system (1.7) is compatible [1]. In order to prove the sufficiency of the compatibility of (1.7) for linearization, Lie showed that the system (1.7) can be linearized and the resulting linear system belongs to a class of special type of linear systems which can be reduced to a linear third-order ODE. Thus the three solutions  $(z, w)$ ,  $(z_1, w_1)$  and  $(z_2, w_2)$  of the system (1.7) can be found by solving this linear third-order ODE. Finally, these solutions can be used as a basis for constructing the linearizing transformations by solving the quadratures

$$\begin{aligned}\frac{\phi_x}{\phi} &= z - z_1, & \frac{\phi_y}{\phi} &= w_1 - w, \\ \frac{\psi_x}{\psi} &= z - z_2, & \frac{\psi_y}{\psi} &= w_2 - w.\end{aligned}\tag{1.9}$$

The aim of this paper is to investigate the linearization of second order ODEs using  $\lambda$ -symmetries. The outline of the paper is as follows. The alternative proof of Lie's approach for linearization of second order ODEs is provided in Section 2. This section also presents a new  $\lambda$ -symmetry linearization criteria which provides an alternative approach for constructing the linearization transformations. The relationship between  $\lambda$ -symmetries and the first integrals of ODEs play the key role in the proving the results of Section 2. A familiarity with standard results about the theory of  $\lambda$ -symmetries is assumed and the reader is referred to the basic works [14, 15, 16] on  $\lambda$ -symmetries. In Section 3, we apply the new approach to linearize the ODEs of the form (1.2) with  $F_3 = F_2 = 0$ . Section 4 consists of examples illustrating the application of the new approach to linearize ODEs of the form (1.2) with  $F_3 \neq 0$  or  $F_2 \neq 0$ .

## 2 Alternative proof of Lie's linearization approach & new $\lambda$ -symmetry linearization criteria

In this section, an alternative proof of Lie's approach to linearization of second order ODEs is presented. It is noticed that the relation between the  $\lambda$ -symmetries and the first integrals provides a direct method to derive both the Lie's conditions and the quadratures. In addition, a  $\lambda$ -symmetry criteria for linearization via a point transformation is stated. This criteria provides a new approach for constructing the linearizing transformations, utilizing  $\lambda$ -symmetries.

The two first integrals

$$I_1 = u_t, \quad I_2 = u - t u_t \quad (2.10)$$

of the free particle equation (1.3) take the form

$$I_1 = \frac{\psi_x + \psi_y y'}{\phi_x + \phi_y y'}, \quad I_2 = \psi - \phi \left( \frac{\psi_x + \psi_y y'}{\phi_x + \phi_y y'} \right), \quad (2.11)$$

when expressed in the new variables  $x$  and  $y(x)$  defined by equation (1.4). It follows that all linearizable equations (1.2) should have the two first integrals (2.11). Hence, the relationship between the first integrals and the  $\lambda$ -symmetries [15, 16] implies that all linearizable equations (1.2) should have the two  $\lambda$ -symmetries equivalent to the canonical pairs  $(\frac{\partial}{\partial y}, \lambda_1)$  and  $(\frac{\partial}{\partial y}, \lambda_2)$  given by

$$\begin{aligned} \lambda_1 &= -\frac{I_{1y}}{I_{1y'}} = -A y'^2 - (B + w) y' - z \\ \lambda_2 &= -\frac{I_{2y}}{I_{2y'}} = -A y'^2 - \left( B + w - \frac{\phi_y}{\phi} \right) y' - z + \frac{\phi_x}{\phi}, \end{aligned} \quad (2.12)$$

where  $A, B, w$  and  $z$  are given by equation (1.6)

The free particle equation (1.3) possesses the functionally dependent quotient first integral  $I_3 = \frac{I_2}{I_1} = \frac{u-t}{u_t}$  as well. It is worthwhile to mention here that the three triplets of Lie algebras of  $I_1, I_2$  and  $I_3$  which are isomorphic to each other generate the Lie algebra

$sl(3, R)$  of the free particle equation [17]. Similar to the above, it can be seen that all linearizable equations (1.2) should have the third associated  $\lambda$ -symmetry equivalent to the canonical pair  $(\frac{\partial}{\partial y}, \lambda_3)$ , where

$$\lambda_3 = -\frac{I_{3y}}{I_{3y'}} = -A y'^2 - \left(B + w - \frac{\psi_y}{\psi}\right) y' - z + \frac{\psi_x}{\psi}. \quad (2.13)$$

The expressions for  $\lambda_1$ ,  $\lambda_2$  and  $\lambda_3$  can be simplified, using the system (1.5), as

$$\begin{aligned} \lambda_1 &= -F_3 y'^2 - (F_2 - w) y' - z \\ \lambda_2 &= -F_3 y'^2 - \left(F_2 - w - \frac{\phi_y}{\phi}\right) y' - z + \frac{\phi_x}{\phi} \\ \lambda_3 &= -F_3 y'^2 - \left(F_2 - w - \frac{\psi_y}{\psi}\right) y' - z + \frac{\psi_x}{\psi}. \end{aligned} \quad (2.14)$$

Employing the definition of  $\lambda$ -symmetry [14], the canonical pair  $(\frac{\partial}{\partial y}, \lambda_1)$  leads to the following system for  $w$  and  $z$

$$\begin{aligned} w_y &= -w^2 + F_2 w + F_3 z + \frac{\partial F_3}{\partial x} - F_1 F_3, \\ w_x - z_y &= 2 w z - 2 F_3 F + \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \\ z_x &= z^2 - F_1 z - F w + \frac{\partial F}{\partial y} + F F_2. \end{aligned} \quad (2.15)$$

Similarly, it follows by applying the definition of  $\lambda$ -symmetry for  $\lambda_2$  and  $\lambda_3$ , along with using the system (2.15), that the transformations  $\phi$  and  $\psi$  can be given by finding two linearly independent non-constant solutions of the following over-determined system

$$\begin{aligned} S_{yy} + (2w - F_2) S_y + F_3 S_x &= 0 \\ S_{xy} + w S_x - z S_y &= 0 \\ S_{xx} + (F_1 - 2z) S_x - F S_y &= 0 \end{aligned} \quad (2.16)$$

Based on the above discussion we have obtained the following two  $\lambda$ -symmetry criteria for linearization via point transformations.

**Theorem 2.1.** *A scalar second-order ODE (1.1) is linearizable via point transformations (1.4) if and only if it has the cubic in first derivative form (1.2) with the  $\lambda$ -symmetries equivalent to the canonical pair  $(\frac{\partial}{\partial y}, \lambda_1)$  for  $\lambda_1 = -F_3 y'^2 - (F_2 - w) y' - z$  and the transformations  $\phi$  and  $\psi$  satisfying equation (2.16), where  $w$  and  $z$  are auxiliary functions.*

*Proof.* The proof in one direction is given in the preceding discussion, so we prove the other way.

Assume that equation (1.2) admits the  $\lambda$ -symmetries equivalent to the canonical pair  $(\frac{\partial}{\partial y}, \lambda_1)$  for  $\lambda_1 = -F_3 y'^2 - (F_2 - w)y' - z$ . Then, the system (2.15) for  $w$  and  $z$  is obtained using the definition of  $\lambda$ -symmetry. Since the transformations  $\phi$  and  $\psi$  satisfy equation (2.16), the compatibility of the system (2.16), i.e.  $S_{xy} = S_{yx}$ ,  $S_{xyx} = S_{xxy}$  and  $S_{xyy} = S_{yyx}$ , leads to the system

$$\begin{aligned} \phi_y \left( -z_x + z^2 - F_1 z - Fw + \frac{\partial F}{\partial y} + FF_2 \right) - \phi_x \left( -2z_y - zw - w_x + \frac{\partial F_1}{\partial y} + FF_3 \right) &= 0 \\ \phi_x \left( w_y + w^2 - F_2 w - F_3 z - \frac{\partial F_3}{\partial x} + F_1 F_3 \right) - \phi_y \left( 2w_x - zw + z_y - \frac{\partial F_2}{\partial x} + FF_3 \right) &= 0. \end{aligned} \quad (2.17)$$

Substituting the system (2.15) into the system (2.17) and noting that the Jacobian

$J = \phi_x \psi_y - \phi_y \psi_x \neq 0$ , one finds

$$w_x + z_y = \frac{1}{3} \left( \frac{\partial F_2}{\partial x} + \frac{\partial F_1}{\partial y} \right). \quad (2.18)$$

Finally, system (2.15) and equation (2.18) are equivalent to the Lie's conditions (1.7) for linearizable equations and so the compatibility,  $w_{xy} = w_{yx}$  and  $z_{xy} = z_{yx}$ , of the Lie's conditions (1.7) leads to the invariant equations (1.8).  $\square$

**Corollary 2.2.** *A scalar second-order ODE (1.1) is linearizable via point transformations if and only if it has the cubic in first derivative form (1.2) with the  $\lambda$ -symmetries equivalent to the canonical pair  $(\frac{\partial}{\partial y}, \lambda_1)$  for  $\lambda_1 = -F_3 y'^2 - (F_2 - w)y' - z$ , where  $w$  and  $z$  are auxiliary functions satisfying the equation  $w_x + z_y = \frac{1}{3} \left( \frac{\partial F_2}{\partial x} + \frac{\partial F_1}{\partial y} \right)$ .*

**Remark 2.3.** The compatibility of the system (2.16) is guaranteed by both the system (2.15) and equation (2.18), i.e.  $S_{xy} = S_{yx}$ ,  $S_{xyx} = S_{xxy}$  and  $S_{xyy} = S_{yyx}$ . Hence for each  $w$  and  $z$  given by solving (2.15) and (2.18), one can construct transformations by solving the over-determined system (2.16). This provides a new approach for constructing

the linearizing transformations whose implementation requires finding only one solution of the system (1.7). In comparison, the standard implementation of Lie's linearization approach involves solving Lie's quadratures (1.9) which requires finding three solutions of the system (1.7).

Finally, in order to obtain an alternative proof of Lie's quadratures (1.9) using  $\lambda$ -symmetries, it is noticed that  $\lambda_1$ ,  $\lambda_2$  and  $\lambda_3$  given by (2.14) can be written as

$$\begin{aligned}\lambda_1 &= -F_3 y'^2 - (F_2 - w) y' - z, \\ \lambda_2 &= -F_3 y'^2 - (F_2 - w_1) y' - z_1, \\ \lambda_3 &= -F_3 y'^2 - (F_2 - w_2) y' - z_2,\end{aligned}\tag{2.19}$$

where  $w_1 = w + \frac{\phi_y}{\phi}$ ,  $z_1 = z - \frac{\phi_x}{\phi}$ ,  $w_2 = w + \frac{\psi_y}{\psi}$  and  $z_2 = z - \frac{\psi_x}{\psi}$ . Therefore, using the definition of  $\lambda$ -symmetry shows that  $(z, w)$ ,  $(z_1, w_1)$  and  $(z_2, w_2)$  are the three solutions for the system (1.7). This completes the alternative proof of Lie's approach.

### 3 Linearization of the ODE's of the form (1.2) with

$$F_3 = F_2 = 0$$

The non-linear second order ODEs of the form

$$y'' + F_1(x, y)y' + F(x, y) = 0\tag{3.20}$$

satisfy the Lie-Tressé linearization criteria (1.8) if and only if

$$F_1(x, y) = a(x)y + b(x), \quad F(x, y) = \frac{(a(x))^2}{9}y^3 + \frac{1}{3} \left( \frac{da(x)}{dx} + a(x)b(x) \right) y^2 + c(x)y + d(x)$$

Here we consider the class of nonlinear equations

$$\frac{d^2y}{dx^2} + (a(x)y + b(x)) \frac{dy}{dx} + \frac{1}{9} (a(x))^2 y^3 + \frac{1}{3} \left( \frac{da(x)}{dx} + a(x)b(x) \right) y^2 + c(x)y + d(x) = 0\tag{3.21}$$



where  $a(x) \neq 0$ .

Solving the system (1.7) gives the three solutions  $(w, z)$ ,  $(w_1, z_1)$  and  $(w_2, z_2)$

$$\begin{aligned} w(x, y) &= \frac{a(x)g_1(x)}{ya(x)g_1(x)-3g_1'(x)}, & z(x, y) &= \frac{1}{3} \frac{9a(x)g_1''(x)-(9a'(x)+6ya(x)^2)g_1'(x)+y^2a(x)^3g_1(x)}{a(x)(ya(x)g_1(x)-3g_1'(x))}, \\ w_1(x, y) &= \frac{a(x)g_2(x)}{ya(x)g_2(x)-3g_2'(x)}, & z_1(x, y) &= \frac{1}{3} \frac{9a(x)g_2''(x)-(9a'(x)+6ya(x)^2)g_2'(x)+y^2a(x)^3g_2(x)}{a(x)(ya(x)g_2(x)-3g_2'(x))}, \\ w_2(x, y) &= \frac{a(x)g_3(x)}{ya(x)g_3(x)-3g_3'(x)}, & z_2(x, y) &= \frac{1}{3} \frac{9a(x)g_3''(x)-(9a'(x)+6ya(x)^2)g_3'(x)+y^2a(x)^3g_3(x)}{a(x)(ya(x)g_3(x)-3g_3'(x))}, \end{aligned} \quad (3.22)$$

where  $g_i(x), i = 1, 2, 3$ , are the three linearly independent solutions of the linear third order ODE

$$Y'''(x) + \left(b(x) - 2\frac{a'(x)}{a(x)}\right)Y''(x) - \left(\frac{a''(x)}{a(x)} - 2\frac{a'(x)^2}{a(x)^2} + b(x)\frac{a'(x)}{a(x)} - c(x)\right)Y'(x) + \frac{d(x)a(x)}{3}Y(x) = 0. \quad (3.23)$$

Now, in order to construct the linearizing transformation using Lie's linearization approach, one should solve the quadratures (1.9) by considering the three solutions  $(w, z)$ ,  $(w_1, z_1)$  and  $(w_2, z_2)$ . However, our approach requires utilizing any one the three solutions  $(w, z)$ ,  $(w_1, z_1)$ ,  $(w_2, z_2)$  in order to determine two non-constant solutions of the over-determined system (2.16) which yield the linearization transformations.

As an illustration, we consider ODEs (3.21) with  $d(x) = 0$ . For this case  $g_1(x) = 1$  is a constant solution of the ODE (3.23). So equation (3.22) gives the solution  $(w, z) = \left(\frac{1}{y}, \frac{1}{3}ya(x)\right)$  of the system (1.7). Now, solving the over-determined system (2.16) gives the linearizing point transformations

$$\phi(x, y) = \frac{1}{3} \int a(x)h_1(x)dx - \frac{h_1(x)}{y}, \quad \psi(x, y) = \frac{1}{3} \int a(x)h_2(x)dx - \frac{h_2(x)}{y}. \quad (3.24)$$

where  $h_i(x), i = 1, 2$ , are the two linearly independent solutions of the linear second order ODE

$$H''(x) + b(x)H'(x) + c(x)H(x) = 0. \quad (3.25)$$

Finally, since  $u(t) = c_1t + c_2$  is the general solution of the free particle equation  $u_{tt} = 0$ ,

the general solution of the ODE (3.21) can be given as

$$\frac{1}{3} \int a(x)h_2(x)dx - \frac{h_2(x)}{y} = c_1 \left( \frac{1}{3} \int a(x)h_1(x)dx - \frac{h_1(x)}{y} \right) + c_2. \quad (3.26)$$

**Examples 3.1.** We consider the ODE (3.21) with  $a(x) = 3, b(x) = 0$  and  $c(x) = 0$  that gives the modified Emden equation

$$y'' + 3yy' + y^3 = 0, \quad (3.27)$$

for which the ODE (3.25) reduces to

$$H''(x) = 0. \quad (3.28)$$

Hence the linearizing point transformations can be written using (3.24) as

$$\phi(x, y) = x - \frac{1}{y}, \quad \psi(x, y) = \frac{x^2}{2} - \frac{x}{y}. \quad (3.29)$$

Finally the general solution can be written using (3.26) as

$$y(x) = \frac{2x + c_1}{x^2 + c_1x + c_2}. \quad (3.30)$$

**Examples 3.2.** For  $a(x) = 3k, b(x) = b$  and  $c(x) = \frac{b^2}{4}$ , ODE (3.21) gives the Liénard type equation

$$y'' + (b + 3ky)y' + k^2y^3 + bky^2 + \frac{b^2}{4}y = 0, \quad (3.31)$$

and the ODE (3.25) reduces to

$$H''(x) + bH'(x) + \frac{b^2}{4}H(x) = 0. \quad (3.32)$$

So the linearizing point transformations can be stated, using (3.24), as

$$\phi(x, y) = \frac{2ky+b}{bky}e^{-\frac{b}{2}x}, \quad \psi(x, y) = \frac{2bkxy+4ky+b^2x}{b^2ky}e^{-\frac{b}{2}x}, \quad (3.33)$$

which lead to the general solution via (3.26) as

$$y(x) = \frac{b^2(c_1 - x)}{2bkx + 4k - 2c_1bk + c_2b^2ke^{\frac{b}{2}x}}. \quad (3.34)$$

## 4 Examples of linearization of the ODEs of the form (1.2) with $F_3 \neq 0$ or $F_2 \neq 0$

In this section, we illustrate the application of our approach to obtain linearization transformations for ODEs that are cubic or quadratic in the first derivative.

**Examples 4.1.** As the first example, we consider the nonlinear ODE

$$y'' - \frac{2}{x+y}y'^2 - \frac{1}{x+y}y' = 0 \quad (4.35)$$

which satisfies the Lie-Tressé linearization test (1.8). The three solutions  $(w, z)$ ,  $(w_1, z_1)$  and  $(w_2, z_2)$  of the Lie's conditions (1.7) are

$$\begin{aligned} w(x, y) &= -\frac{1}{x+y}, & z(x, y) &= 0, \\ w_1(x, y) &= \frac{x}{y(x+y)}, & z_1(x, y) &= 0, \\ w_2(x, y) &= \frac{x}{(x+2y)(x+y)}, & z_2(x, y) &= -\frac{2(x+y)}{x(x+2y)}. \end{aligned} \quad (4.36)$$

Our approach requires only one solution of the Lie's conditions (1.7). Solving the over-determined system (2.16), by considering only  $(w, z)$ , gives the two non-constant solutions which results in the point transformations

$$\phi(x, y) = y, \quad \psi(x, y) = x(x + 2y). \quad (4.37)$$

that linearize the ODE (4.35) to the free particle equation  $u_{tt} = 0$ . So the general solution of the ODE (4.35) is

$$x(x + 2y) = c_1y + c_2. \quad (4.38)$$

It is worth mentioning that any of the other solutions  $(w_1, z_1)$  and  $(w_2, z_2)$  of the Lie's conditions (1.7) can be used to derive different linearizing point transformations that will lead to the same general solution.

**Examples 4.2.** The nonlinear ODE

$$xy'' - y'^3 - y' = 0 \quad (4.39)$$

satisfies the Lie-Tressé linearization test (1.8). A solution  $(w, z)$  of the Lie's conditions (1.7) is

$$w(x, y) = \frac{1}{y}, \quad z(x, y) = 0. \quad (4.40)$$

Now, solving the over-determined system (2.16) gives the two non-constant solutions which gives rise to the point transformation

$$\phi(x, y) = \frac{1}{y}, \quad \psi(x, y) = y + \frac{x^2}{y} \quad (4.41)$$

that linearize the ODE (4.43) to the free particle equation  $u_{tt} = 0$ . So the general solution of the ODE (4.43) is obtained as

$$x^2 + y^2 = c_1 + c_2 y. \quad (4.42)$$

**Examples 4.3.** In this example, we find the general solution of the nonlinear ODE

$$y'' - \frac{1}{x}y'^3 + \frac{2y}{y^2 - 1}y'^2 - \frac{1}{x}y' = 0 \quad (4.43)$$

via linearization transformations. ODE (4.43) satisfies the Lie-Tressé linearization test (1.8) and a solution  $(w, z)$  of the Lie's conditions (1.7) is

$$w(x, y) = \frac{3y^2 - 3}{y^3 - 3y}, \quad z(x, y) = 0. \quad (4.44)$$

Now, solving the over-determined system (2.16) gives the two non-constant solutions which gives the point transformation

$$\phi(x, y) = \frac{1}{y(y^2 - 3)}, \quad \psi(x, y) = \frac{(y^3 - 3y + 2)\ln(y - 1) + (2 + 3y - y^3)\ln(y + 1) + 3x^2 - y^2}{y(y^2 - 3)} \quad (4.45)$$

that linearize the ODE (4.43) to the free particle equation  $u_{tt} = 0$ . Hence the general solution of the ODE (4.43) can be written as

$$(y^3 - 3y + 2)\ln(y - 1) + (2 + 3y - y^3)\ln(y + 1) + 3x^2 - y^2 = c_1 + c_2 y(y^2 - 3). \quad (4.46)$$

## 5 Conclusion

The question of linearization of second order ODEs is investigated employing  $\lambda$ -symmetries. The relationship between  $\lambda$ -symmetries and the first integrals plays an important role and provides a direct method to derive both of Lie's conditions (1.7) and quadratures (1.9). The relationship further leads to a  $\lambda$ -symmetry criteria for linearization via point transformations. This criteria provides a new approach for constructing the linearizing transformations whose implementation requires finding only one solution of the Lie's conditions (1.7). In comparison, the standard implementation of Lie's linearization approach involves solving Lie's quadratures (1.9) which requires finding three solutions of the Lie's conditions (1.7).

It is expected that the relationship between  $\lambda$ -symmetries and the first integrals of ODEs can play a significant role in deriving new linearization criteria for higher order ODEs.

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## References

- [1] Lie S, Klassifikation und Integration von gewöhnlichen Differentialgleichungen zwischen  $x, y$ , die eine Gruppe von Transformationen gestatten. III, Archiv for Matematik og Naturvidenskab 8 (Kristiania, 1883), 371–458 [reprinted in Lie's Gesammelte Abhandlungen 5, 1924, paper XIV, 362–427].

- [2] Tresse A M, Sur les Invariants Differentiels des Groupes Continus de Transformations, Acta Mathematica 1894, 18, 1 – 88.
- [3] Ibragimov N H and Mahomed F M, Ordinary differential equations, CRC Handbook of Lie Group Analysis of Differential Equations, Vol. 3. Ibragimov N H (ed.), CRC Press: Boca Raton, 1996, 191.
- [4] Mahomed F M, Symmetry group classification of ordinary differential equations: Survey of some results, Math. Meth. Appl. Sci., 2007, 30, 1995 - 2012.
- [5] Qadir Asghar, Linearization: Geometric, Complex, and Conditional, Journal of Applied Mathematics, Vol. 2012, Article ID 303960, 30 pages, doi: 10.1155/2012/303960.
- [6] Ibragimov N H, *A practical course in differential equations and mathematical modelling* ALGA Publications, Bleking Institute of technology Karlskrona, Sweden & , (2006).
- [7] Grissom C, Thompson G and Wilkens G, Linearization of second-order ordinary differential equations via Cartans equivalence method, Journal of Differential Equations, 1989, 77, 1.
- [8] Ibragimov N H and Magri F, Geometric proof of Lies linearization theorem, Nonlinear Dynamics, 2004, 36, 41 – 46.
- [9] Mahomed F M and Qadir Asghar, Invariant Linearization Criteria for Systems of Cubically Nonlinear Second-Order Ordinary Differential Equations, Journal of Nonlinear Mathematical Physics, Vol. 16, No. 3, 2009, 283 – 298.
- [10] Ali S, Mahomed F M and Qadir Asghar, Linearizability criteria for systems of two second-order differential equations by complex methods, Nonlinear Dynamics, 2011, 66, 77 – 88, doi: 10.1007/s11071-010-9912-2.

- [11] Mahomed F M and Qadir Asghar, Conditional Linearizability Criteria for Third Order Ordinary Differential Equations, Journal of Nonlinear Mathematical Physics, Vol. 15, Sup 1, 2008, 124 - 133.
- [12] Sarlet W, Mahomed F M and Leach P G L, Symmetries of non-linear differential equations and linearization, Journal of Physics A: Mathematical and General, 1987, 20, 277.
- [13] Mahomed F M and Leach P G L, The Lie algebra  $sl(3, R)$  and linearization, Quaestiones Mathematicae, 1989, 12, 121.
- [14] Muriel C and Romero J L 2001 New methods of reduction for ordinary differential equations IMA J. Appl. Math. 66 111 - 25
- [15] Muriel C and Romero J L 2008 Integrating factors and  $\lambda$ -symmetries J. Nonl. Math. Phys. 15 290 - 9
- [16] Muriel C and Romero J L 2009 First integrals, integrating factors and  $\lambda$ -symmetries of second-order differential equations, J. Phys. A: Math. Theor. 42 365207
- [17] Leach P G L and Mahomed F M, Maximal subalgebra associated with a first integral of a system possessing  $sl(3, R)$  symmetry, Journal of Mathematical Physics, 1988, 29, 1807.